A master equation for quantum systems driven by Poisson white noise

1991 J. Phys. A: Math. Gen. 24 L1021

(http://iopscience.iop.org/0305-4470/24/17/010)

J. Phys. A: Math. Gen. 24 (1991) L1021–L1024. Printed in the UK

LETTER TO THE EDITOR

A master equation for quantum systems driven by Poisson white noise

J Łuczka† and M Niemiec‡

† Department of Theoretical Physics, Silesian University, 40-007 Katowice, Poland
‡ Institute of Biophysics, Silesian Academy of Medicine, 40-952 Katowice 6, Poland

Received 10 May 1991, in final form 11 July 1991

Abstract. An evolution equation for a reduced statistical operator of quantum systems driven by Poisson white noise is derived. It is applied to a simple system and compared with a counterpart driven by Gaussian white noise.

The influence of noise on systems has been extensively studied (for recent reviews see [1, 2]). When studying effects of noise on a system, one should distinguish between the cases of external and internal as well as classical and quantum noises. In this letter we consider a quantum system driven by white Poisson noise. A source of such noise may be external and it can be treated as classical or internal noise and it can have a quantum nature [3]. External Poisson noise can be generated, for example, by voltaic impulses, flashes of light, laser impulses, etc, with prescribed stochastic characteristics of point processes [4, 5]. The quantum character of Poisson noise can be related to any quantum point process that occurs with random intensity and randomly in time, e.g. optical signals as a result of spontaneous emission of excited atoms, quantum jumps, emission of photoelectrons, etc [3, 5].

Let

$$H = H_0 + z(t)V \tag{1}$$

be the Hamiltonian of a quantum system, where H_0 and V are Hermitian operators and $z(t)$ is a real Poisson white noise [6–8]

$$z(t) = \sum_{i=1}^{N(t)} \xi_i \delta(t - t_i) - \nu\langle\xi\rangle. \tag{2}$$

The process $N(t)$ is a Poisson counting process [5, 9] with probability

$$P(N(t) = n) = (\nu t)^n \, e^{-\nu t}/n!$$

(ν is a mean number of peaks per unit time). The random variables ξ_i are independent of each other and distributed with the same probability density $P(\xi)$, $\langle\xi\rangle$ is a mean value of ξ_i over $P(\xi)$ and the random times t_i are uniformly distributed in the interval $(0, t)$. The process $z(t)$ has zero mean value

$$\langle z(t)\rangle = 0$$

and is delta correlated (white noise)

$$\langle z(t)z(s)\rangle = \nu\langle\xi^2\rangle\delta(t - s).$$

There are several additional reasons why it is worth studying the system described by equations (1) and (2). For example, compare properties of classical systems and their quantum counterparts driven by the same noise; compare behaviours of a quantum system driven by different noises; compare a quantum system driven by noise (2) and a periodically kicked system (random and deterministic delta peaks and quantum chaos problems [10]).

Our aim is to obtain an evolution equation for a reduced statistical operator

$$\rho_s(t) = \langle \rho(t) \rangle_z \tag{3}$$

which is the average over all realizations of $z(t)$ (indicated by the subscript z in (3)) of the density operator $\rho(t)$ that obeys the Liouville–von Neumann equation

$$\dot{\rho}(t) = -(\mathrm{i}/\hbar)[H, \rho(t)]. \tag{4}$$

Taking the average (4) we get

$$\dot{\rho}_s(t) = -(i/\hbar)[H_0, \rho_s(t)] - (i/\hbar)\langle z(t)[V, \rho(t)]\rangle_z. \tag{5}$$

Equation (5) is not closed. To proceed further, we use the Klyatskin–Tatarsky formula [11] adapted to the process (2). It has the form

$$\langle z(t) R[z]\rangle_z = -\nu\langle\xi\rangle\langle R[z]\rangle_z + \nu \int_{-\infty}^{\infty} \mathrm{d}\xi\, P(\xi) \int_0^{\xi} \mathrm{d}\eta \left\langle \exp\left[\eta \frac{\delta}{\delta z(t)}\right] R[z]\right\rangle_z \tag{6}$$

where $R[z]$ is an arbitrary functional of the process (2). In our case $R[z] = [V, \rho(t)]$ and $\rho(t) = \rho[z(t)]$ is a functional of $z(t)$ via equations (4) and (1). The functional derivative

$$\frac{\delta}{\delta z(t)} \rho(t) = -(\mathrm{i}/\hbar)[V, \rho(t)] \tag{7}$$

and hence

$$\exp\left[\eta \frac{\delta}{\delta z(t)}\right] \rho(t) = \mathrm{e}^{-(\mathrm{i}/\hbar)\eta V} \rho(t)\, \mathrm{e}^{(\mathrm{i}/\hbar)\eta V}. \tag{8}$$

Using (6)–(8) in (5), one obtains

$$\dot{\rho}_s(t) = -(\mathrm{i}/\hbar)[H_0, \rho_s(t)] + (\mathrm{i}/\hbar)\nu\langle\xi\rangle[V, \rho_s(t)] - \nu\rho_s(t) + \nu \int_{-\infty}^{\infty} \mathrm{d}\xi\, P(\xi)\, \mathrm{e}^{-(\mathrm{i}/\hbar)\xi V} \rho_s(t)\, \mathrm{e}^{(\mathrm{i}/\hbar)\xi V} \tag{9}$$

where we have used the identity

$$\int_0^{\xi} \mathrm{d}\eta [V, \mathrm{e}^{-(\mathrm{i}/\hbar)\eta V} \rho\, \mathrm{e}^{(\mathrm{i}/\hbar)\eta V}] = \mathrm{i}\hbar(\mathrm{e}^{-(\mathrm{i}/\hbar)\xi V} \rho\, \mathrm{e}^{-(\mathrm{i}/\hbar)\xi V} - \rho)$$

which is valid for an arbitrary operator ρ.

Equation (9) is a desired equation for a reduced statistical operator $\rho_s(t)$. It can be generalized for the case

$$H = H_0 + \sum_k z_k(t) V_k$$

where $z_k(t)$ are independent Poisson white noises. For comparison, a quantum system driven by Gaussian white noise $(z(t) \to \Gamma(t))$

$$\langle \Gamma(t) \rangle = 0 \qquad \langle \Gamma(t)\Gamma(s) \rangle = 2D\delta(t-s)$$

is described by the equation [12]

$$\dot{\rho}_s(t) = -(\mathrm{i}/\hbar)[H_0, \rho_s(t)] - (D/\hbar^2)[V, [V, \rho_s(t)]] \tag{10}$$

with the diffusion coefficient D.

As an illustration, let us consider a two-level system perturbed by random peaks in the x-direction

$$H = \hbar\omega_0 S^z + \hbar z(t) S^x \tag{11}$$

where S^i $(i = x, y, z)$ are the spin $S = \frac{1}{2}$ operators. From (9) it follows that

$$\dot{x} = -\omega_0 y \qquad \dot{y} = \omega_0 x + \Omega z - \gamma y \qquad \dot{z} = -\Omega y - \gamma z \tag{12}$$

where

$$x = \langle S^x \rangle \qquad y = \langle S^y \rangle \qquad z = \langle S^z \rangle$$

and

$$\Omega = \nu \int_{-\infty}^{\infty} \mathrm{d}\xi (\xi - \sin \xi) P(\xi) \tag{13}$$

$$\gamma = \nu \int_{-\infty}^{\infty} \mathrm{d}\xi (1 - \cos \xi) P(\xi). \tag{14}$$

A system similar to (11) but driven by Gaussian white noise $\Gamma(t)$ is described by the set of equations

$$\dot{x} = -\omega_0 y \qquad \dot{y} = \omega_0 x - Dy \qquad \dot{z} = -Dz. \tag{15}$$

Equations (12) and (15) represent rotations (with frequencies ω_0 and Ω) and damping (with the rate γ or D) of the y and z components. If $\Omega = 0$ then (12) and (15) have the same form. Otherwise, an effective field of strength proportional to Ω is produced by $z(t)$ (but not by $\Gamma(t)$). For example, if ξ_i takes a single value, $\xi_i = \xi_0$, then $P(\xi) = \delta(\xi - \xi_0)$ and $\Omega \neq 0$. If $P(\xi)$ is an exponential distribution, $P(\xi) = \alpha\Theta(\xi)\exp(-\alpha\xi)$, where $\alpha > 0$ and Θ is the Heaviside function, then $\Omega \neq 0$ as well. If $P(\xi)$ is a Gaussian distribution, $P(\xi) \sim \exp(-a\xi^2)$, $a > 0$, then $\Omega = 0$. In both cases (12) and (15), the noises $z(t)$ and $\Gamma(t)$ induce damping and relaxation processes appear in the system. Their action is equivalent to an action of a heat bath (surroundings) and the system behaves as an open system. In general, it is not a rule. For example for a quantum harmonic oscillator with $V = x$ (where x is a position variable) the noises $z(t)$ or $\Gamma(t)$ do not induce damping but on the contrary act as a pump of energy to the system.

An application of the theory presented here to less trivial quantum models will be considered elsewhere.

This work is supported in part by the Committee of Science Research. The authors would like to thank a referee for remarks which allowed them to improve the presentation.

References

[1] Moss F and McClintock P V E (ed) 1989 *Noise in Nonlinear Dynamical Systems: Theory, Experiment, Simulation* vol I–III (Cambridge: Cambridge University Press)

[2] Doering C R, Brand H R and Ecke R E (ed) 1989 *Proc. Workshop on External Noise and its Interaction with Spatial Degree of Freedom* (*J. Stat. Phys.* **54**)

[3] Schenzle A 1989 *J. Stat. Phys.* **54** 1243

[4] Fuliński A 1987 *Phys. Lett.* **126A** 84

[5] Solodov A W and Solodov A A 1988 *Statistical Dynamics of Systems with Point Processes* (Moscow: Nauka) (in Russian)

[6] Feynman R P and Hibbs A R 1965 *Quantum Mechanics and Path Integrals* (New York: McGraw-Hill)

[7] Klyatskin V I 1975 *Statistical Description of Dynamical Systems with Fluctuating Parameters* (Moscow: Nauka) (in Russian)

[8] Rodriguez M A, Pesquera L, San Miguel M and Sancho J M 1985 *J. Stat. Phys.* **40** 669

[9] Papoulis A 1965 *Probability, Random Variables, and Stochastic Processes* (New York: McGraw-Hill)

[10] Lan B L and Fox R F 1991 *Phys. Rev.* A **43** 646

[11] Klyatskin V I and Tatarsky V I 1973 *Teor. Mat. Fiz.* **17** 273

[12] Łuczka J 1991 *Czech. J. Phys.* **41** 289